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Complex Maxwell groups in the description of evanescent photons

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Abstract. For inhomogeneous electromagnetic waves in isotropic media the operator evolution solutions of Maxwell equations are obtained. These solutions are functions of the complex spatial variable. In the case of homogeneous waves an evolution operator is associated with a set of right-handed and left-handed generalized helices. In this set the helices of straight circular cylinders are geodetic lines. It is shown that one of the branches of the evolution operators is generated by traceless operators and that it correspond to standing evanescent waves. These evolution operators are elements of the group $SL(2, C)$. For this Maxwell group a new parametrization is introduced and appropriate composition laws are derived. The introduced parametrization is compared with the known Fedorov parametrization.

1. Introduction

In [1] the operator solutions of the one-dimensional tensor Helmholtz equations for electromagnetic and acoustical fields in isotropic media were investigated. These solutions are evolutional and have an infinite set of branches which is divided into two groups of operators: (a) with traces not equal to zero and (b) with traces equal to zero. The first group characterizes the polarization evolution of running waves and the second characterizes the evolution of standing waves which are elliptically polarized in the general case. In [1–4] some features of the obtained and similar solutions were highlighted.

(i) They belong to general operator solutions or to be exact to the sets of such solutions. The operator approach turns out to be useful for solving numerous problems of light and sound propagation in chiral, dispersive and inhomogeneous moving media. In contrast to the usual methods of integration of motion equations on the coordinate base, it enables us to establish clear and laconic relationships between the emission characteristics and material tensors. It may be promising for the investigation of inhomogeneous waves in problems of total internal reflection. What is more such an approach is closely connected with group theory which enables us in many cases to investigate the symmetries of wave equations and their solutions without straightforward integration. Operator solutions are not needed in the use of partial waves. This typical feature of the approach as applied to complex modulated media makes it possible to avoid fairly cumbersome and tiresome calculations. Connection of the concepts 'partial wave' and 'evolution operator' results from the known mathematical relations between the partial and the fundamental solutions of systems of ordinary differential equations. The fundamental solution describes the complex motion of field vectors and photons corresponding to them. This conforms to Galilei's principle of motion superposition known in mechanics. To investigate the vector motions even in

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homogeneous isotropic media it is necessary to use modern methods of differential geometry and topology. The key role in such a description is played by three-dimensional vectors of wavenormals and two-dimensional surfaces of wavefronts. Immersed into the three-dimensional space the fronts have their own metrics. Schrödinger in his Nobel report [5] examined wave refraction in a layered medium and paid attention to the primordial importance of wavefronts in comparison with rays. Validity of the conclusion made in [5] is confirmed not only by progress in quantum mechanics but by the works on phase defects and dislocations of optical wavefronts, on topological phases [6] and others as well.

(ii) The operator solutions in explicit form involve the operators of isometries (reflection and rotation) which characterize symmetries of fields and possible sources related with them. These isometries confirm the existence of the conservation laws and some arbitrariness of the choice of models of numerical and spacetime sets. The existence of the infinite set of helicoidal elliptical solutions which relate to standing waves shows itself by the presence of photons and antiphotons (Pauli, Weisskopf) in nature. This symmetry is closely connected with the charge symmetry. In [1] for curves on helicoid surfaces the general coordinate dependences of their curvatures, torsions and Darboux vectors were found. This family of right-handed and left-handed helices, describing a spatial distribution of circular polarized light field, is a family of geodetic lines. In the case of a fixed value of the field vector in an initial point for an evolution operator there corresponds a grid of such geodetic lines of the straight circular cylinder. Solutions discussed in [1] can be treated as varieties of solitons (Helmholtz solitons?). Soliton solutions can be the solutions of the linear non-dispersive equations [7]. Helicoidal standing waves may be applied to study properties of the space-time continuum and its discrete modifications by modern methods of information theory [8], in the investigations of the topological features of vortex motions [9, 10], in topology of the wavefronts of acoustical waves [11] and polaritons [14].

(iii) The solutions of the tensor dispersion equations give the generators of continuous groups to be the branches of the square root $\sqrt{\varepsilon\mu \boldsymbol{I}} = \sqrt{\varepsilon\mu(1 - \boldsymbol{n}\otimes\boldsymbol{n})}$. Here $\boldsymbol{I} = 1 - \boldsymbol{n}\otimes\boldsymbol{n}$ is the projector to a wavefront surface. Appropriate global operators (Cauchy operators) in concentrated mathematical form express the essence of the Huyghens principle for polarized waves [1, 3, 4].

(iv) Many of the evolution operators, almost without any changes, are applied to the very important class of inhomogeneous or evanescent waves [12–21]. For this purpose it is necessary to complexify the wavenormal and, consequently, the wavefront as well. In the literature there are no references to the operator complex description (without division of fields into partial waves) of inhomogeneous electromagnetic waves except in some of our works where, however, the possibilities of applying this approach to evanescent waves are not displayed properly.

In this paper, we, to indicate some new features of the operator solutions as applied to inhomogeneous waves, carry out a classification and establish the composition laws for parameters of the continuous Maxwell groups which are generated by traceless operators. In fact we discover the new parametrization of the group $SL(2, C)$ being the symmetry group of optical evolution operators. We establish the connection of this parameterization with the known Fedorov parametrization [22, 23] of the group $SO(3, C)$ isomorphic to $SL(2, C)$. The subsequent sections of the paper are devoted to consideration of the stated problems.

2. Evolution solutions and standing inhomogeneous waves

In [1] we show that the solutions of Maxwell equations

$$\nabla^{\times}\boldsymbol{E} = \mathrm{i}k\mu\boldsymbol{H} \qquad \nabla^{\times}\boldsymbol{H} = -\mathrm{i}k\varepsilon\boldsymbol{E}$$

($k = \omega/c$ and $\boldsymbol{\nabla}^{\times}_{ik} = -\boldsymbol{\nabla}^{\times}_{ki} = e_{ilk}\boldsymbol{\nabla}_l$) for plane waves in a homogeneous isotropic medium with the permittivity ε and the permeability μ can be represented in the evolutional form

$$\boldsymbol{H}_\tau(z) = \exp[\mathrm{i}kN(z - z_0)]\boldsymbol{H}_\tau(z_0) \tag{1}$$

where $z = \boldsymbol{n} \cdot \boldsymbol{r}$, $\boldsymbol{n}$ is a real wave normal and N is the second-rank refractive index tensor. It was shown that N yields the relations

$$N^2 = \varepsilon\mu I \qquad I = -\boldsymbol{n}^{\times 2} = 1 - \boldsymbol{n} \otimes \boldsymbol{n} \tag{2}$$

which is the tensor generalization of the scalar Maxwell relation $n^2 = \varepsilon\mu$, where two different cases are possible

$$N = \pm\sqrt{\varepsilon\mu}(I - 2\boldsymbol{S} \otimes \boldsymbol{C}) \qquad N_{\mathrm{t}} = 0 \tag{3}$$

$$\boldsymbol{S} \cdot \boldsymbol{C} = 1 \qquad \boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0 \tag{4}$$

and

$$N = \pm\sqrt{\varepsilon\mu} I \qquad N_{\mathrm{t}} = \pm 2\sqrt{\varepsilon\mu}. \tag{5}$$

In (3) and (5) and hereafter, a subscript t is used to indicate trace. In the case of absorption or amplification of homogeneous waves for the evolution operator (1) together with the initial field vector there is a corresponding set of generalized conic helices [24–26].

In this section we generalize, at first, the expressions (1)–(5) for inhomogeneous waves with the complex wave normal $\boldsymbol{n}$ ($\boldsymbol{n}^* \neq \boldsymbol{n}$) and, second, using the spectral expansions of operators, we establish the fact that the operator N in form (3) describes standing waves. In the next section we carry out group analysis of evolution operators involved in (1) and derive composition laws. The main result here is that operators N in form (3) generate evolution operators belonging to the group $SL(2, C)$.

As is known [12, 13], inhomogeneous waves in the general case are described by the complex wave normal $\boldsymbol{n}$. The complex vector $\boldsymbol{n}$ may be represented in the form

$$\boldsymbol{n} = \boldsymbol{a} + \mathrm{i}\boldsymbol{b} \tag{6}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are real vectors. Without loss of generality the wave normal $\boldsymbol{n}$ (6) may be taken to be normalized

$$\boldsymbol{n}^2 = \boldsymbol{n} \cdot \boldsymbol{n} = 1.$$

Then $\boldsymbol{a}$ and $\boldsymbol{b}$ obey the conditions

$$\boldsymbol{a}^2 - \boldsymbol{b}^2 = 1 \qquad \boldsymbol{a} \cdot \boldsymbol{b} = 0.$$

Other forms of representation of $\boldsymbol{n}$ are possible, in particular [20]

$$\boldsymbol{n} = \cosh\psi\,\boldsymbol{n}_1 + \mathrm{i}\sinh\psi\,\boldsymbol{n}_2$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are real unit vectors and $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$, ψ is a real parameter.

Supposing that all the field vectors $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{B}$ depend on the complex coordinate $\xi = \boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{a} \cdot \boldsymbol{r} + \mathrm{i}\boldsymbol{b} \cdot \boldsymbol{r}$ and repeating the calculations stated in [1] we conclude that the spatial evolution of $\boldsymbol{H}$ is given by a formula which is analogous to (1)

$$\boldsymbol{H}_\tau(\xi) = \exp[\mathrm{i}kN(\xi - \xi_0)]\boldsymbol{H}_\tau(\xi_0) \tag{7}$$

where

$$\boldsymbol{n} \cdot \boldsymbol{H}_\tau(\xi) = \boldsymbol{n} \cdot \boldsymbol{H}_\tau(\xi_0) = 0. \tag{8}$$

Equation (2) remains valid and its solutions are (3) and (5). The difference is that the projective operator I in these formulae is no longer real and Hermitian. In view of (8) the evolution of $\boldsymbol{H}_\tau$ takes place in the complex plane orthogonal to $\boldsymbol{n}$. This plane can be characterized by two vectors

$$\boldsymbol{e}_1 = -\mathrm{i}\sqrt{\frac{\boldsymbol{b}^2}{\boldsymbol{a}^2}}\boldsymbol{a} + \sqrt{\frac{\boldsymbol{a}^2}{\boldsymbol{b}^2}}\boldsymbol{b} \qquad \boldsymbol{e}_2 = \frac{[\boldsymbol{ab}]}{\sqrt{\boldsymbol{a}^2\boldsymbol{b}^2}} \tag{9}$$

which belong to it. Here $[\boldsymbol{ab}]$ denotes the vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$. It is not difficult to verify that

$$\boldsymbol{e}_1\cdot\boldsymbol{n} = \boldsymbol{e}_2\cdot\boldsymbol{n} = \boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 0 \qquad \boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 1 \qquad \boldsymbol{n} = [\boldsymbol{e}_1\boldsymbol{e}_2]$$
$$I = \boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \boldsymbol{e}_2\otimes\boldsymbol{e}_2. \tag{10}$$

It is evident that the vector $\boldsymbol{H}_\tau(\xi)$ can be written as a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ (9).

Now we consider a particular case when

$$\boldsymbol{S} = \boldsymbol{e}^{(-)} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_2) \qquad \boldsymbol{C} = \boldsymbol{e}^{(+)} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \mathrm{i}\boldsymbol{e}_2) \tag{11}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are determined by (9). In this case $\boldsymbol{S}$ and $\boldsymbol{C}$ are circular vectors and $\boldsymbol{S}^2 = \boldsymbol{C}^2 = \boldsymbol{e}^{(-)^2} = \boldsymbol{e}^{(+)^2} = 0$ and $\boldsymbol{S}\cdot\boldsymbol{C} = \boldsymbol{e}^{(-)}\cdot\boldsymbol{e}^{(+)} = 1$. Straightforward substitution of (11) in (3) in consideration of (10) and $\boldsymbol{e}_2\otimes\boldsymbol{e}_1 - \boldsymbol{e}_1\otimes\boldsymbol{e}_2 = [\boldsymbol{e}_1\boldsymbol{e}_2]^\times$ [12, 13] leads to

$$N_{\mathrm{vers}} = \pm\mathrm{i}\sqrt{\varepsilon\mu}\,\boldsymbol{n}^\times$$

where N_{vers} is a rotation operator (so called *versor*). It is evident that this operator is among the traceless operators.

To understand better the character of the varying vector $\boldsymbol{H}_\tau(\xi)$ as the coordinate ξ changes we shall find a spectral expansion of the evolution operator

$$\Omega(\xi - \xi_0) = \exp[\mathrm{i}kN(\xi - \xi_0)] \tag{12}$$

involved in (7). We shall suppose in the process that the operator N is in form (3). It is known that any three-dimensional operator X acting in a two-dimensional subspace can be represented in the spectral form

$$X = \lambda_1\rho_1 + \lambda_2\rho_2 \tag{13}$$

supposing that its non-zero eigenvalues λ_1 and λ_2 are not equal. Here in (13) ρ_1 and ρ_2 are projective operators

$$\rho_1^2 = \rho_1 \qquad \rho_2^2 = \rho_2 \qquad \rho_1\rho_2 = \rho_2\rho_1 = 0.$$

These projective operators can be found by using the relations

$$\rho_1 = \frac{X - \lambda_2 I}{\lambda_1 - \lambda_2} \qquad \rho_2 = \frac{X - \lambda_1 I}{\lambda_2 - \lambda_1} \tag{14}$$

where I is a projective operator of the two-dimensional subspace (in the case under consideration $I = -\boldsymbol{n}^{\times^2}$). The eigenvalues of X are determined from the characteristic equation

$$|X - \lambda I| = 0.$$

Since $|X| = \frac{1}{2}[(X_{\mathrm{t}})^2 - (X^2)_{\mathrm{t}}]$ and $I_{\mathrm{t}} = 2$ then

$$|X - \lambda I| = \tfrac{1}{2}\{[(X - \lambda I)_{\mathrm{t}}]^2 - [(X - \lambda I)^2]_{\mathrm{t}}\} = \lambda^2 - X_{\mathrm{t}}\lambda + |X|$$

and the characteristic equation takes the form

$$\lambda^2 - X_{\mathrm{t}}\lambda + |X| = 0. \tag{15}$$

We can find any regular function of X as

$$f(X) = f(\lambda_1)\rho_1 + f(\lambda_2)\rho_2.$$

For N (3) we immediately obtain from (15) and (14) that

$$\lambda_1 = \pm\sqrt{\varepsilon\mu} \qquad \lambda_2 = \mp\sqrt{\varepsilon\mu}$$

and

$$\rho_1 = \boldsymbol{I} - \boldsymbol{S} \otimes \boldsymbol{C} \qquad \rho_2 = \boldsymbol{S} \otimes \boldsymbol{C}. \tag{16}$$

Then the spectral expansion of $\Omega(\xi - \xi_0)$ (12) is

$$\Omega(\xi - \xi_0) = \exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)]\rho_1 + \exp[\mp \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)]\rho_2 \tag{17}$$

where ρ_1 and ρ_2 are given by formulae (16).

There always exists such a vector $\boldsymbol{R}$ ($\boldsymbol{R} \cdot \boldsymbol{n} = 0$) that $\boldsymbol{C} \cdot \boldsymbol{R} = 0$. It is evident that $[\boldsymbol{S}\boldsymbol{R}] \neq 0$. Thus vectors $\boldsymbol{S}$ and $\boldsymbol{R}$ form a basis in the plane orthogonal to $\boldsymbol{n}$. $\boldsymbol{S}$ and $\boldsymbol{R}$ are eigenvectors of the projective operators ρ_1 and ρ_2

$$\rho_1\boldsymbol{S} = 0 \qquad \rho_1\boldsymbol{R} = \boldsymbol{R} \qquad \rho_2\boldsymbol{S} = \boldsymbol{S} \qquad \rho_2\boldsymbol{R} = 0. \tag{18}$$

Since the set $\boldsymbol{S}$, $\boldsymbol{R}$ is a basis then the expansion of the initial field vector $\boldsymbol{H}_\tau(\xi_0)$ is possible

$$\boldsymbol{H}_\tau(\xi_0) = H_R\boldsymbol{R} + H_S\boldsymbol{S}.$$

In such a case and in view of (17) and (18) equation (7) is rewritten as

$$\boldsymbol{H}_\tau(\xi) = H_R\boldsymbol{R}\exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)] + H_S\boldsymbol{S}\exp[\mp \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)]. \tag{19}$$

Remembering that $\xi = \boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{a} \cdot \boldsymbol{r} + \mathrm{i}\boldsymbol{b} \cdot \boldsymbol{r}$ we obtain another expression for $\boldsymbol{H}_\tau(\xi)$

$$\begin{aligned}\boldsymbol{H}_\tau(\boldsymbol{r}) = H_R\boldsymbol{R}&\exp[\mp k\sqrt{\varepsilon\mu}\boldsymbol{b} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)]\exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}\boldsymbol{a} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)] \\ &+H_S\boldsymbol{S}\exp[\mp k\sqrt{\varepsilon\mu}\boldsymbol{b} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)]\exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}\boldsymbol{a} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)].\end{aligned} \tag{20}$$

From (19) and (20) it clearly follows that the traceless operators N (3) generate solutions describing in essence standing inhomogeneous waves which are superposition inhomogeneous waves running in the opposite directions along $\boldsymbol{a}$. The amplitudes of such waves are characterized by the vectors $\boldsymbol{S}$ and $\boldsymbol{R}$ and increase or decrease in the direction along $\boldsymbol{b}$.

As for the second branch of solutions (5) it is clear that these are simply running waves

$$\begin{aligned}&\Omega(\xi - \xi_0) = \exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)]I \\ &\boldsymbol{H}_\tau(\xi) = \exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)]\boldsymbol{H}_\tau(\xi_0).\end{aligned}$$

Returning to the example with $\boldsymbol{S}$ and $\boldsymbol{C}$ (11) we see that the vector $\boldsymbol{C} = \boldsymbol{e}^{(+)}$ is in the capacity of $\boldsymbol{R}$ here ($\boldsymbol{C} \cdot \boldsymbol{R} = \boldsymbol{C} \cdot \boldsymbol{C} = 0$). Therefore, in this case

$$\boldsymbol{H}_\tau(\xi) = H^{(+)}\boldsymbol{e}^{(+)}\exp[\pm \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)] + H^{(-)}\boldsymbol{e}^{(-)}\exp[\mp \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)]$$

where $H^{(+)}$ and $H^{(-)}$ are the expansion coefficients of the initial vector $\boldsymbol{H}_\tau(\xi_0)$ in the basis $\boldsymbol{e}^{(+)}$ and $\boldsymbol{e}^{(-)}$.

3. Group $SL(2, C)$ of symmetries of the operator evolution solutions

In this section we ascertain to which groups the evolution operators generated by N (3) and (5) belong. First, we consider families of the traceless operators N (3). The following facts should be pointed out here. In the first place all the operators Ω (17) act in two-dimensional complex space. Second, determinants of these operators equal one because they are exponentials of traceless operators. This fact can be tested by a straightforward calculation too. Third, any transformation Ω (17) can be characterized by the parameters $\boldsymbol{S}$, $\boldsymbol{C}$ and $\eta \equiv \pm \mathrm{i}k\sqrt{\varepsilon\mu}(\xi - \xi_0)$. The product $\Omega(\eta', \boldsymbol{S}', \boldsymbol{C}')\Omega(\eta'', \boldsymbol{S}'', \boldsymbol{C}'')$ has unit determinant and again acts in two-dimensional complex space. It can be represented as an exponential of some traceless operator. Now, we shall show that any traceless operator L ($\boldsymbol{n}L = L\boldsymbol{n} = 0$, $L_{\mathrm{t}} = 0$) can be written in the general form

$$L = \eta(I - 2\boldsymbol{S} \otimes \boldsymbol{C}) \tag{21}$$

where $\boldsymbol{S}$ and $\boldsymbol{C}$ are some complex vectors ($\boldsymbol{S} \cdot \boldsymbol{n} = \boldsymbol{C} \cdot \boldsymbol{n} = 0$, $\boldsymbol{S} \cdot \boldsymbol{C} = 1$), η is a complex value. We can choose an orthonormal basis in the complex space, for example, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ given by (9), and write L as

$$L = z_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + z_2\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + z_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_1$$

where z_1, z_2, z_3 are complex parameters. Let $\eta = (z_1^2 - z_2 z_3)^{1/2}$. Then

$$L = \eta L_0 \tag{22}$$

where

$$L_0 = \frac{1}{\eta}[z_1(\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2) + z_2\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + z_3\boldsymbol{e}_2 \otimes \boldsymbol{e}_1].$$

Finally, let us represent L_0 as

$$L_0 = I - L_1 \tag{23}$$

where

$$L_1 = I - L_0 = \left(1 - \frac{z_1}{\eta}\right)\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \left(1 + \frac{z_1}{\eta}\right)\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 - \frac{z_2}{\eta}\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 - \frac{z_3}{\eta}\boldsymbol{e}_2 \otimes \boldsymbol{e}_1.$$

It is easy to see that $|L_1| = 0$ and $(L_1)_{\mathrm{t}} = 2$. This means that L_1 is a diad $2\boldsymbol{S} \otimes \boldsymbol{C}$ with $\boldsymbol{S}$ and $\boldsymbol{C}$ obeying $\boldsymbol{S} \cdot \boldsymbol{C} = 1$. Thus, from (23) and (22) representation (21) follows. So

$$\Omega(\eta', \boldsymbol{S}', \boldsymbol{C}')\Omega(\eta'', \boldsymbol{S}'', \boldsymbol{C}'') = \Omega(\eta, \boldsymbol{S}, \boldsymbol{C})$$

with some new parameters η, $\boldsymbol{S}$, $\boldsymbol{C}$. From the aforesaid, it follows that the evolution operators $\Omega = \exp[\eta(I - 2\boldsymbol{S} \otimes \boldsymbol{C})]$ form the group, η, $\boldsymbol{S}$ and $\boldsymbol{C}$ being parameters of this group. If η, $\boldsymbol{S}$ and $\boldsymbol{C}$ do not obey any conditions then the number of independent real parameters is $2 + 6 + 6 = 14$. However, the conditions (4) decrease this number from 14 to eight. Finally, operators (21) are invariant under transformations $\boldsymbol{S} \to \alpha\boldsymbol{S}$ and $\boldsymbol{C} \to (1/\alpha)\boldsymbol{C}$, where α is any complex number. This circumstance decreases the number of independent parameters from eight to six. Thus the group under consideration is in fact six-parameter. It is evident that it is the group $SL(2, C)$ of unimodular operators in two-dimensional complex space, which is a complexification of the group $SU(2)$ [27]. It remains for us to establish a composition law

$$\eta = \eta(\eta', \eta'', \boldsymbol{S}', \boldsymbol{S}'', \boldsymbol{C}', \boldsymbol{C}'')$$
$$\boldsymbol{S} = \boldsymbol{S}(\eta', \eta'', \boldsymbol{S}', \boldsymbol{S}'', \boldsymbol{C}', \boldsymbol{C}'')$$
$$\boldsymbol{C} = \boldsymbol{C}(\eta', \eta'', \boldsymbol{S}', \boldsymbol{S}'', \boldsymbol{C}', \boldsymbol{C}'').$$

For this purpose we derive the product of two Ω-operators

$$
\begin{aligned}
\Omega(\eta', \boldsymbol{S}', \boldsymbol{C}')\Omega(\eta'', \boldsymbol{S}'', \boldsymbol{C}'') &\equiv \Omega'\Omega'' \\
&= [\mathrm{e}^{\eta'}(I - \boldsymbol{S}' \otimes \boldsymbol{C}') + \mathrm{e}^{-\eta'}\boldsymbol{S}' \otimes \boldsymbol{C}'][\mathrm{e}^{\eta''}(I - \boldsymbol{S}'' \otimes \boldsymbol{C}'') + \mathrm{e}^{-\eta''}\boldsymbol{S}'' \otimes \boldsymbol{C}''] \\
&= \mathrm{e}^{\eta'+\eta''}I - 2\,\mathrm{e}^{\eta''}\sinh\eta'\boldsymbol{S}' \otimes \boldsymbol{C}' - 2\,\mathrm{e}^{\eta'}\sinh\eta''\boldsymbol{S}'' \otimes \boldsymbol{C}'' \\
&\quad +4\sinh\eta'\sinh\eta''(\boldsymbol{C}' \cdot \boldsymbol{S}'')\boldsymbol{S}' \otimes \boldsymbol{C}''
\end{aligned}
$$

and reduce it to spectral form to compare with expansion (17), rewritten here as

$$
\Omega(\eta, \boldsymbol{S}, \boldsymbol{C}) = \mathrm{e}^{\eta}(I - \boldsymbol{S} \otimes \boldsymbol{C}) + \mathrm{e}^{-\eta}\boldsymbol{S} \otimes \boldsymbol{C}. \tag{24}
$$

Taking into account $(\boldsymbol{C}' \cdot \boldsymbol{S}'')(\boldsymbol{C}'' \cdot \boldsymbol{S}') = 1 - [\boldsymbol{S}'\boldsymbol{S}''] \cdot [\boldsymbol{C}'\boldsymbol{C}'']$ we have

$$
\begin{aligned}
(\Omega'\Omega'')_{\mathrm{t}} &\equiv 2\Delta \\
&= 2[\cosh(\eta' + \eta'') - 2\sinh\eta'\sinh\eta''[\boldsymbol{S}'\boldsymbol{S}''] \cdot [\boldsymbol{C}'\boldsymbol{C}'']] \qquad |\Omega'\Omega''| = 1.
\end{aligned} \tag{25}
$$

Then the characteristic equation (15) takes the form

$$
\lambda^2 - 2\Delta\lambda + 1 = 0
$$

and has the following solutions

$$
\lambda_1 = \Delta + \sqrt{\Delta^2 - 1} \qquad \lambda_2 = \Delta - \sqrt{\Delta^2 - 1}.
$$

In view of $\lambda_1 = \mathrm{e}^{\eta}$ and $\lambda_2 = \mathrm{e}^{-\eta}$ one can obtain the expression for η

$$
\eta = \ln(\Delta + \sqrt{\Delta^2 - 1}) = \operatorname{arccosh}\Delta.
$$

Furthermore, we find a projector ρ_2 which is the diad $\boldsymbol{S} \otimes \boldsymbol{C}$

$$
\begin{aligned}
\rho_2 = \frac{\Omega'\Omega'' - \lambda_1 I}{\lambda_1 - \lambda_2} = -\frac{1}{2\sqrt{\Delta^2 - 1}}&[(\mathrm{e}^{\eta'+\eta''} - \Delta - \sqrt{\Delta^2 - 1})I - 2\,\mathrm{e}^{\eta''}\sinh\eta'\boldsymbol{S}' \otimes \boldsymbol{C}' \\
&-2\,\mathrm{e}^{\eta'}\sinh\eta''\boldsymbol{S}'' \otimes \boldsymbol{C}'' + 4\sinh\eta'\sinh\eta''(\boldsymbol{C}' \cdot \boldsymbol{S}'')\boldsymbol{S}' \otimes \boldsymbol{C}''].
\end{aligned} \tag{26}
$$

On the other hand, we can obtain another expression for ρ_2 valid for $\boldsymbol{C}'' \cdot \boldsymbol{S}' \neq 0$. It is clear that $\rho_2\boldsymbol{S}' = \beta\boldsymbol{S}$ and $\boldsymbol{C}''\rho_2 = \gamma\boldsymbol{C}$, where β and γ are some coefficients of proportionality. Then $\rho_2\boldsymbol{S}' \otimes \boldsymbol{C}''\rho_2 = \beta\gamma\boldsymbol{S} \otimes \boldsymbol{C} = \beta\gamma\rho_2$ and $(\rho_2\boldsymbol{S}' \otimes \boldsymbol{C}''\rho_2)_{\mathrm{t}} = \beta\gamma$. In addition, we have

$$
(\rho_2\boldsymbol{S}' \otimes \boldsymbol{C}''\rho_2)_{\mathrm{t}} = (\rho_2^2\boldsymbol{S}' \otimes \boldsymbol{C}'')_{\mathrm{t}} = (\rho_2\boldsymbol{S}' \otimes \boldsymbol{C}'')_{\mathrm{t}} = \boldsymbol{C}''\rho_2\boldsymbol{S}'. \tag{27}
$$

Therefore,

$$
\rho_2 = \boldsymbol{S} \otimes \boldsymbol{C} = \frac{\rho_2\boldsymbol{S}' \otimes \boldsymbol{C}''\rho_2}{\boldsymbol{C}''\rho_2\boldsymbol{S}'} = \rho_2\boldsymbol{S}' \otimes \frac{\boldsymbol{C}''\rho_2}{\boldsymbol{C}''\rho_2\boldsymbol{S}'}. \tag{28}
$$

The part $\rho_2\boldsymbol{S}'$ we can identify as the vector $\boldsymbol{S}$ and the part $\boldsymbol{C}''\rho_2/\boldsymbol{C}''\rho_2\boldsymbol{S}'$ as the vector $\boldsymbol{C}$

$$
\boldsymbol{S} = \rho_2\boldsymbol{S}' \qquad \boldsymbol{C} = \frac{\boldsymbol{C}''\rho_2}{\boldsymbol{C}''\rho_2\boldsymbol{S}'}. \tag{29}
$$

From (26) it follows that

$$
\rho_2\boldsymbol{S}' = -\frac{1}{2\sqrt{\Delta^2 - 1}}[(\Delta - \mathrm{e}^{\eta'-\eta''} - \sqrt{\Delta^2 - 1})\boldsymbol{S}' - 2\,\mathrm{e}^{\eta'}\sinh\eta''(\boldsymbol{C}'' \cdot \boldsymbol{S}')\boldsymbol{S}''] \tag{30}
$$

$$
\boldsymbol{C}''\rho_2 = -\frac{1}{2\sqrt{\Delta^2 - 1}}[-2\,\mathrm{e}^{\eta''}\sinh\eta'(\boldsymbol{C}'' \cdot \boldsymbol{S}')\boldsymbol{C}' + (\Delta - \mathrm{e}^{-(\eta'-\eta'')} - \sqrt{\Delta^2 - 1})\boldsymbol{C}''] \tag{31}
$$

$$
\boldsymbol{C}''\rho_2\boldsymbol{S}' = \frac{1}{2\sqrt{\Delta^2 - 1}}(\boldsymbol{C}'' \cdot \boldsymbol{S}')(\mathrm{e}^{\eta'+\eta''} - \Delta + \sqrt{\Delta^2 - 1}). \tag{32}
$$

Summing up we obtain the following composition law ($\boldsymbol{C}'' \cdot \boldsymbol{S}' \neq 0$)

$$
\begin{aligned}
&\eta = \operatorname{arccosh}\Delta \\
&\boldsymbol{S} = \frac{1}{2\sqrt{\Delta^2-1}}[\sqrt{\Delta^2-1} - \Delta + \mathrm{e}^{\eta'-\eta''})\boldsymbol{S}' + 2\,\mathrm{e}^{\eta'}\sinh\eta''(\boldsymbol{C}''\cdot\boldsymbol{S}')\boldsymbol{S}''] \\
&\boldsymbol{C} = \frac{1}{(\boldsymbol{C}''\cdot\boldsymbol{S}')[\mathrm{e}^{\eta'+\eta''} - \Delta + \sqrt{\Delta^2-1}]} \\
&\qquad\qquad \times[2\,\mathrm{e}^{\eta''}\sinh\eta'(\boldsymbol{C}''\cdot\boldsymbol{S}')\boldsymbol{C}' + (\sqrt{\Delta^2-1} - \Delta + \mathrm{e}^{-(\eta'-\eta'')})\boldsymbol{C}'']
\end{aligned} \tag{33}
$$

where the value Δ is determined by (25). The case $\boldsymbol{C}'' \cdot \boldsymbol{S}' = 0$ will be considered later separately.

Now we analyse cases when the formulae (33) are significantly simplified. In particular, these are cases when the square root $\sqrt{\Delta^2-1}$ can be completely extracted. For example, when $[\boldsymbol{S}'\boldsymbol{S}''] \cdot [\boldsymbol{C}'\boldsymbol{C}''] = 0$ (or $(\boldsymbol{C}' \cdot \boldsymbol{S}'')(\boldsymbol{C}'' \cdot \boldsymbol{S}') = 1$) the square root is $\sqrt{\Delta^2-1} = \sinh(\eta'+\eta'')$. The foregoing condition means that either $\boldsymbol{S}' \parallel \boldsymbol{S}''$ or $\boldsymbol{C}' \parallel \boldsymbol{C}''$. In this case

$$
\begin{aligned}
&\eta = \eta' + \eta'' \\
&\boldsymbol{S} = \frac{1}{\sinh(\eta'+\eta'')}[\mathrm{e}^{-\eta''}\sinh\eta'\boldsymbol{S}' + \mathrm{e}^{\eta'}\sinh\eta''(\boldsymbol{C}''\cdot\boldsymbol{S}')\boldsymbol{S}''] \\
&\boldsymbol{C} = \frac{1}{(\boldsymbol{C}''\cdot\boldsymbol{S}')\sinh(\eta'+\eta'')}[\mathrm{e}^{\eta''}\sinh\eta'(\boldsymbol{C}''\cdot\boldsymbol{S}')\boldsymbol{C}' + \mathrm{e}^{-\eta'}\sinh\eta''\boldsymbol{C}''].
\end{aligned}
$$

Furthermore, if $\boldsymbol{S}' = \boldsymbol{S}''$ and $\boldsymbol{C}' = \boldsymbol{C}''$ then

$$
\eta = \eta' + \eta'' \qquad \boldsymbol{S} = \boldsymbol{S}' = \boldsymbol{S}'' \qquad \boldsymbol{C} = \boldsymbol{C}' = \boldsymbol{C}''
$$

and then again the latter composition law is evident.

The following case when (33) is further simplified by $[\boldsymbol{S}'\boldsymbol{S}''] \cdot [\boldsymbol{C}'\boldsymbol{C}''] = 1$ (or $(\boldsymbol{C}' \cdot \boldsymbol{S}'')(\boldsymbol{C}'' \cdot \boldsymbol{S}') = 0$). To use formulae (33) we must assume that $\boldsymbol{C}' \cdot \boldsymbol{S}'' = 0$ but $\boldsymbol{C}'' \cdot \boldsymbol{S}' \neq 0$. Then $\sqrt{\Delta^2-1} = \sinh(\eta'-\eta'')$ and

$$
\begin{aligned}
&\eta = \eta' - \eta'' \\
&\boldsymbol{S} = \frac{1}{\sinh(\eta'-\eta'')}[\sinh(\eta'-\eta'')\boldsymbol{S}' + \mathrm{e}^{\eta'}\sinh\eta''(\boldsymbol{C}''\cdot\boldsymbol{S}')\boldsymbol{S}''] \\
&\boldsymbol{C} = \boldsymbol{C}'.
\end{aligned} \tag{34}
$$

To obtain the composition law for the case when $\boldsymbol{C}'' \cdot \boldsymbol{S}' = 0$ and $\boldsymbol{C}' \cdot \boldsymbol{S}'' \neq 0$ we must repeat the general derivation (27)–(32) for $\boldsymbol{S}$ and $\boldsymbol{C}$ but multiply the operator ρ_2 (26) by $\boldsymbol{S}''$ from the right and by $\boldsymbol{C}'$ from the left. Then

$$
\begin{aligned}
&\eta = \eta' - \eta'' \\
&\boldsymbol{S} = \boldsymbol{S}' \\
&\boldsymbol{C} = \frac{1}{\sinh(\eta'-\eta'')}[\sinh(\eta'-\eta'')\boldsymbol{C}' + \mathrm{e}^{-\eta'}\sinh\eta''(\boldsymbol{C}'\cdot\boldsymbol{S}'')\boldsymbol{C}''].
\end{aligned} \tag{35}
$$

Interestingly, the composition law is achieved when both $\boldsymbol{C}' \cdot \boldsymbol{S}'' = 0$ and $\boldsymbol{C}'' \cdot \boldsymbol{S}' = 0$. Then from (34) or (35) it follows that

$$
\eta = \eta' - \eta'' \qquad \boldsymbol{S} = \boldsymbol{S}' \qquad \boldsymbol{C} = \boldsymbol{C}'.
$$

Let $\eta' = \eta''$. Then $\Omega(\eta, \boldsymbol{S}, \boldsymbol{C}) = \Omega(0, \boldsymbol{S}, \boldsymbol{C}) = I$ and

$$
\Omega(\eta', \boldsymbol{S}', \boldsymbol{C}')\Omega(\eta', \boldsymbol{S}'', \boldsymbol{C}'') = I
$$

or

$$\Omega(-\eta', \boldsymbol{S}', \boldsymbol{C}') = \Omega(\eta', \boldsymbol{S}'', \boldsymbol{C}'') \tag{36}$$

under the condition that $\boldsymbol{C}' \cdot \boldsymbol{S}'' = 0$ and $\boldsymbol{C}'' \cdot \boldsymbol{S}' = 0$. This means that we can confine the region of values of η by one of the complex half-planes. The obtained result (36) can be easily understood if we consider the evolution of $\boldsymbol{H}_\tau(\xi_0)$ under the action of the operators $\Omega(-\eta', \boldsymbol{S}', \boldsymbol{C}')$ or $\Omega(\eta', \boldsymbol{S}'', \boldsymbol{C}'')$ with application of formula (19). In this case either $\boldsymbol{R} = \boldsymbol{S}''$ or $\boldsymbol{R} = \boldsymbol{S}'$, respectively.

In order for the conditions $\boldsymbol{C}' \cdot \boldsymbol{S}'' = 0$ and $\boldsymbol{C}'' \cdot \boldsymbol{S}' = 0$ to be fulfilled we can choose $\boldsymbol{S}'' = [\boldsymbol{n}\boldsymbol{C}']$ and $\boldsymbol{C}'' = [\boldsymbol{n}\boldsymbol{S}']$. Then from (24) and (36) it follows that

$$\Omega^{-1} = \mathrm{e}^{-\eta}(\boldsymbol{I} - \boldsymbol{S} \otimes \boldsymbol{C}) + \mathrm{e}^{\eta}\boldsymbol{S} \otimes \boldsymbol{C} = \mathrm{e}^{\eta}(\boldsymbol{I} - [\boldsymbol{n}\boldsymbol{C}] \otimes [\boldsymbol{n}\boldsymbol{S}]) + \mathrm{e}^{-\eta}[\boldsymbol{n}\boldsymbol{C}] \otimes [\boldsymbol{n}\boldsymbol{S}]. \tag{37}$$

Making use of (24) and (37) we conclude that

$$\tilde{\Omega}\boldsymbol{n}^{\times} = \mathrm{e}^{\eta}(\boldsymbol{n}^{\times} - \boldsymbol{C} \otimes [\boldsymbol{S}\boldsymbol{n}]) + \mathrm{e}^{-\eta}\boldsymbol{C} \otimes [\boldsymbol{S}\boldsymbol{n}] = \boldsymbol{n}^{\times}\Omega^{-1}$$

or

$$\tilde{\Omega}\boldsymbol{n}^{\times}\Omega = \boldsymbol{n}^{\times}$$

where tilde denotes transposition. Let $\boldsymbol{A}(0)$ and $\boldsymbol{B}(0)$ be arbitrary vectors which do not coincide and orthogonal to $\boldsymbol{n}$ and $\boldsymbol{A}(\eta) = \Omega\boldsymbol{A}(0)$ and $\boldsymbol{B}(\eta) = \Omega\boldsymbol{B}(0)$ be vectors transformed under (24). Then

$$\boldsymbol{A}(\eta)\boldsymbol{n}^{\times}\boldsymbol{B}(\eta) = \boldsymbol{A}(0)\tilde{\Omega}\boldsymbol{n}^{\times}\Omega\boldsymbol{B}(0) = \boldsymbol{A}(0)\boldsymbol{n}^{\times}\boldsymbol{B}(0).$$

So the transformation (24) retains the scalar product $\boldsymbol{A}\boldsymbol{n}^{\times}\boldsymbol{B}$ with a weight $\boldsymbol{n}^{\times}$ as invariant. Making the transition to a particular orthonormal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{n} = \boldsymbol{e}_3$ we have $\boldsymbol{A}\boldsymbol{n}^{\times}\boldsymbol{B} = A_2B_1 - A_1B_2$. In this case the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ have transformation properties like the spinors of the group $SL(2, C)$.

Thus in this section we have in essence introduced a new parametrization of the group $SL(2, C)$. As it turned out, this group is closely connected with the problem of light propagation in isotropic media. As for the group of operators generated by N of type (5) it is clear that this group is isomorphic to the group of complex numbers. We do not consider this trivial question in detail here.

Also in [1] the branches of the square root $\sqrt{1}$ of the three-dimensional unit operator 1 were considered. These are ± 1 and $\pm(1 - 2\boldsymbol{S} \otimes \boldsymbol{C})$ where $\boldsymbol{S} \cdot \boldsymbol{C} = 1$ and appear in the operator solutions of the Christoffel equation for isotropic media when velocities of the transverse and longitudinal elastic waves coincide. In contrast to operators (3) and (5) they have traces ± 3 and ± 1, respectively. Therefore, all the evolution operators of the Christoffel equation generated by $1 - 2\boldsymbol{S} \otimes \boldsymbol{C}$ have determinants not equal to one. These operators form a group but for this group it is impossible to introduce η, $\boldsymbol{S}$, $\boldsymbol{C}$-parametrization as was done for $SL(2, C)$. In fact, both $1 - 2\boldsymbol{S} \otimes \boldsymbol{C}$ and

$$\exp[\eta(1 - 2\boldsymbol{S} \otimes \boldsymbol{C})] = \mathrm{e}^{\eta}(1 - \boldsymbol{S} \otimes \boldsymbol{C}) + \mathrm{e}^{-\eta}\boldsymbol{S} \otimes \boldsymbol{C} \tag{38}$$

have two coinciding eigenvalues. However, the product of exponentials of type (38) has eigenvalues which are non-degenerate in the general case. This means that $\exp[\eta'(1-2\boldsymbol{S}' \otimes \boldsymbol{C}')]\exp[\eta''(1-2\boldsymbol{S}'' \otimes \boldsymbol{C}'')]$ cannot be represented as $\exp[\eta(1-2\boldsymbol{S} \otimes \boldsymbol{C})]$. To confirm this it is sufficient to consider the following simple example. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be an orthonormal basis in three-dimensional space and $\boldsymbol{S}' = \boldsymbol{C}' = \boldsymbol{e}_1$ and $\boldsymbol{S}'' = \boldsymbol{C}'' = \boldsymbol{e}_2$. Then

$$1 - 2\boldsymbol{S}' \otimes \boldsymbol{C}' = -\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$
$$1 - 2\boldsymbol{S}'' \otimes \boldsymbol{C}'' = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 - \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

in view of $1 = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$. We have

$$\exp[\eta'(1 - 2\boldsymbol{S}' \otimes \boldsymbol{C}')] = \mathrm{e}^{-\eta'}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \mathrm{e}^{\eta'}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \mathrm{e}^{\eta'}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \tag{39}$$

$$\exp[\eta''(1 - 2\boldsymbol{S}'' \otimes \boldsymbol{C}'')] = \mathrm{e}^{\eta''}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \mathrm{e}^{-\eta''}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \mathrm{e}^{\eta''}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3. \tag{40}$$

The product of exponentials (39) and (40) equals

$$\mathrm{e}^{-(\eta'-\eta'')}\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \mathrm{e}^{\eta'-\eta''}\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \mathrm{e}^{\eta'+\eta''}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3.$$

It is evident that this product has three non-degenerate eigenvalues $\mathrm{e}^{-(\eta'-\eta'')}$, $\mathrm{e}^{\eta'-\eta''}$ and $\mathrm{e}^{\eta'+\eta''}$, and cannot be represented in the form (38) with any η, $\boldsymbol{S}$, $\boldsymbol{C}$. Thus, for the group of acoustical evolution operators (38) it is impossible to introduce η, $\boldsymbol{S}$, $\boldsymbol{C}$-parametrization. For this group there are not enough values of η, $\boldsymbol{S}$, $\boldsymbol{C}$ to parametrize it. This circumstance is directly connected with the fact that operators (38) act in three-dimensional space in contrast to operators (24) acting in two-dimensional subspace. For the case of acoustical evolution solutions further more detailed investigations are needed.

4. Relationships with the group $SO(3, C)$

To express a finite transformation O of the group $SO(3, C)$ Fedorov has proposed [22] using the complex three-dimensional vector $\boldsymbol{q}$ for a parametrization of this group

$$O(\boldsymbol{q}) = \frac{1 + \boldsymbol{q}^{\times}}{1 - \boldsymbol{q}^{\times}} = 1 + 2\frac{\boldsymbol{q}^{\times} + \boldsymbol{q}^{\times^2}}{1 + \boldsymbol{q}^2}.$$

An evident advantage of this method of parametrization over other ways (for example, by Eulerian angles) is in the very simple composition law: if $O(\boldsymbol{q}) = O(\boldsymbol{q}')O(\boldsymbol{q}'')$ then

$$\boldsymbol{q} = \frac{\boldsymbol{q}' + \boldsymbol{q}'' + [\boldsymbol{q}'\boldsymbol{q}'']}{1 - \boldsymbol{q}' \cdot \boldsymbol{q}''}. \tag{41}$$

The group $SL(2, C)$ is isomorphic to $SO(3, C)$ and therefore can be parametrized by the vector $\boldsymbol{q}$ [22, 28] as well as with the same composition law. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ be an orthonormal basis in complex two-dimensional space (for example, (9)). Then, any unimodular operator Ω can be represented in the form

$$\begin{gathered}\Omega = \alpha_0\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \beta_0\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \gamma_0\boldsymbol{e}_2 \otimes \boldsymbol{e}_1 + \delta_0\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \\ |\Omega| = \alpha_0\delta_0 - \beta_0\gamma_0 = 1.\end{gathered} \tag{42}$$

In [28] the connections between α_0, β_0, γ_0, δ_0 and the components q_1, q_2, q_3 of $\boldsymbol{q}$ were established

$$q_1 = -\mathrm{i}\frac{\beta_0 + \gamma_0}{\alpha_0 + \delta_0} \qquad q_2 = \frac{\gamma_0 - \beta_0}{\alpha_0 + \delta_0} \qquad q_3 = -\mathrm{i}\frac{\alpha_0 - \delta_0}{\alpha_0 + \delta_0} \tag{43}$$

and, conversely,

$$\alpha_0 = \pm\frac{1 + q_3}{\sqrt{1 + \boldsymbol{q}^2}} \qquad \beta_0 = \pm\mathrm{i}\frac{q_1 + \mathrm{i}q_2}{\sqrt{1 + \boldsymbol{q}^2}} \qquad \gamma_0 = \pm\mathrm{i}\frac{q_1 - \mathrm{i}q_2}{\sqrt{1 + \boldsymbol{q}^2}} \qquad \delta_0 = \pm\frac{1 - q_3}{\sqrt{1 + \boldsymbol{q}^2}}. \tag{44}$$

The two-valuedness in (44) is caused by the fact that both Ω and $-\Omega$ satisfy the condition of unimodularity.

It is quite natural to find connections between the $\boldsymbol{q}$-parameter and η, $\boldsymbol{S}$ and $\boldsymbol{C}$-parameters introduced in section 3. For this purpose we represent (42) and (44) in the spectral form using formulae (14) and (15) (without loss of generality we choose in (44) the upper signs)

$$\Omega_t = \frac{2}{\sqrt{1+\boldsymbol{q}^2}} \qquad |\Omega| = 1$$

$$\lambda^2 - \frac{2}{\sqrt{1+\boldsymbol{q}^2}}\lambda + 1 = 0$$

$$\lambda_1 = \frac{1-\mathrm{i}\sqrt{\boldsymbol{q}^2}}{\sqrt{1+\boldsymbol{q}^2}} \qquad \lambda_2 = \frac{1+\mathrm{i}\sqrt{\boldsymbol{q}^2}}{\sqrt{1+\boldsymbol{q}^2}}$$

$$\begin{aligned}\rho_2 = \boldsymbol{S}\otimes\boldsymbol{C} = \frac{1}{2\sqrt{\boldsymbol{q}^2}}[(\sqrt{\boldsymbol{q}^2}+q_3)\boldsymbol{e}_1\otimes\boldsymbol{e}_2 \\ +(q_1+\mathrm{i}q_2)\boldsymbol{e}_1\otimes\boldsymbol{e}_2 + (q_1-\mathrm{i}q_2)\boldsymbol{e}_2\otimes\boldsymbol{e}_1 + (\sqrt{\boldsymbol{q}^2}-q_3)\boldsymbol{e}_2\otimes\boldsymbol{e}_2].\end{aligned}$$

Then

$$\begin{aligned}\eta &= \ln\lambda_1 = \ln\frac{1-\mathrm{i}\sqrt{\boldsymbol{q}^2}}{\sqrt{1+\boldsymbol{q}^2}} \\ \boldsymbol{S} &= \rho_2\boldsymbol{e}_1 = \frac{1}{2\sqrt{\boldsymbol{q}^2}}[(\sqrt{\boldsymbol{q}^2}+q_3)\boldsymbol{e}_1 + (q_1-\mathrm{i}q_2)\boldsymbol{e}_2] \\ \boldsymbol{C} &= \frac{\boldsymbol{e}_1\rho_2}{\boldsymbol{e}_1\rho_2\boldsymbol{e}_1} = \frac{1}{\sqrt{\boldsymbol{q}^2}+q_3}[(\sqrt{\boldsymbol{q}^2}+q_3)\boldsymbol{e}_1 + (q_1+\mathrm{i}q_2)\boldsymbol{e}_2].\end{aligned} \tag{45}$$

So in (45) the parameters η, $\boldsymbol{S}$ and $\boldsymbol{C}$ are expressed in terms of the components of $\boldsymbol{q}$. Now we find the inverse transformations. Using formulae (24) and (42) we have

$$\begin{aligned}\alpha_0 &= \boldsymbol{e}_1\Omega\boldsymbol{e}_1 = \mathrm{e}^\eta - 2\sinh\eta\boldsymbol{S}(\boldsymbol{e}_1\otimes\boldsymbol{e}_1)\boldsymbol{C} \\ \beta_0 &= \boldsymbol{e}_1\Omega\boldsymbol{e}_2 = -2\sinh\eta\boldsymbol{S}(\boldsymbol{e}_1\otimes\boldsymbol{e}_2)\boldsymbol{C} \\ \gamma_0 &= \boldsymbol{e}_2\Omega\boldsymbol{e}_1 = -2\sinh\eta\boldsymbol{S}(\boldsymbol{e}_2\otimes\boldsymbol{e}_1)\boldsymbol{C} \\ \delta_0 &= \boldsymbol{e}_2\Omega\boldsymbol{e}_2 = \mathrm{e}^\eta - 2\sinh\eta\boldsymbol{S}(\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\boldsymbol{C}.\end{aligned} \tag{46}$$

Substituting (46) into (43) and taking into account of the fact that $\boldsymbol{I} = \boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \boldsymbol{e}_2\otimes\boldsymbol{e}_2$, $\boldsymbol{S}\cdot\boldsymbol{C} = 1$ we find

$$q_k = \mathrm{i}\tanh\eta\boldsymbol{S}\sigma_k\boldsymbol{C} \qquad k = 1, 2, 3 \tag{47}$$

where

$$\begin{aligned}\sigma_1 &= \boldsymbol{e}_1\otimes\boldsymbol{e}_2 + \boldsymbol{e}_2\otimes\boldsymbol{e}_1 \\ \sigma_2 &= -\mathrm{i}\boldsymbol{e}_1\otimes\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{e}_2\otimes\boldsymbol{e}_1 = \mathrm{i}\boldsymbol{n}^\times \\ \sigma_3 &= \boldsymbol{e}_1\otimes\boldsymbol{e}_1 - \boldsymbol{e}_2\otimes\boldsymbol{e}_2.\end{aligned}$$

It should be noted that the matrix representation of the operators σ_1, σ_2, σ_3 in the concrete particular basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ corresponds to Pauli matrices.

Thus, the obtained formulae (45) and (47) determine the relationships between two different parametrizations of the group $SL(2, C)$: on the one hand, this is Fedorov $\boldsymbol{q}$-parametrization with the composition law (41) and, on the other, this is η, $\boldsymbol{S}$, $\boldsymbol{C}$-parametrization with the composition laws (33) and (35). One can test that these composition laws are compatible.

5. Conclusion

In the present work we have dealt with the generalization of the operator evolution solutions of Maxwell equations for inhomogeneous electromagnetic waves (evanescent waves) in isotropic media. One of the branches of such solutions is described by the traceless operators of refractive indices which are characterized by the complex vectors $\boldsymbol{S}$ and $\boldsymbol{C}$ as in the case of homogeneous waves. These operators are produced by the extraction of a square root from the unit projective operator $I = -\boldsymbol{n}^{\times^2}$ with the complex wave normal $\boldsymbol{n}$. Using spectral expansions for operators we have shown that the traceless operators $I - 2\boldsymbol{S} \otimes \boldsymbol{C}$ are associated with the superposition of inhomogeneous waves running in opposite directions. In fact, such waves are standing evanescent waves. The tracelessness of the refractive index operators with the complex normal points to the existence of evanescent photons and antiphotons in isotropic media. We have shown that the optical evolution operators are elements of the group $SL(2, C)$. For the first time we have introduced the new η, $\boldsymbol{S}$, $\boldsymbol{C}$-parametrization of this group, the parameters η, $\boldsymbol{S}$ and $\boldsymbol{C}$ being contained in the generators $\eta(I - 2\boldsymbol{S} \otimes \boldsymbol{C})$ of the group. We have obtained composition laws for such a parametrization. The group $SL(2, C)$ directly follows from the operator solutions of the electromagnetic field equations and is generated by reflection and rotation operators. We have called this group a Maxwell group of symmetry of operator solutions. We have established relationships between the known Fedorov vector parameterization of the group $SO(3, C)$ and our η, $\boldsymbol{S}$, $\boldsymbol{C}$-parametrization of the group $SL(2, C)$. As for the group of acoustical evolution operators of the type $\exp[\eta(1 - 2\boldsymbol{S} \otimes \boldsymbol{C})]$, it turns out that η, $\boldsymbol{S}$, $\boldsymbol{C}$-parametrization does not suit here. This is a result of the other dimensionality of the space where these operators act.

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